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ON THE GELFOND TWO CONJECTURES

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ABSTRACT. In this paper, we introduce the two conjectures was proposed by A. O. Gelfond. This two conjectures are very important open problems in the area. I will prove these conjectures under assumption weak Schanuel's conjecture.

1. INTRODUCTION

In [1], references to the existence of transcendental numbers go back many centuries. The "transcendental" comes from Leibniz in his 1682 paper where he proved $\sin x$ is not an algebraic function of x. Certainly Leibniz believed that, besides rational and irrational numbers (by "irrational" he meant algebraic irrational numbers in modern terminology), there also exist transcendental numbers. In [2], Liouville proved a fundamental theorem concerning approximations of algebraic numbers by rational numbers in 1853. This theorem gives first example of transcendental numbers.

Theorem 1.1 (J. Liouville, 1853)**.** *If α is algebraic of degree d, then there is a positive constant $C(\alpha)$, i.e. depending only on α, such that for all rationals $\frac{p}{q}$,*

$$\left|\alpha - \frac{p}{q}\right| > \frac{C(\alpha)}{q^d}.$$

From this theorem, we can find explicit examples of transcendental numbers.

Corollary 1.2. *The number*

$$\sum_{n=0}^{\infty} \frac{1}{2^{n!}}$$

is transcendental number.

In [3], there appeared Hermite's epoch-making memoir entitled *Sur la fonction exponentielle* in which he established the transcendence of e, the natural base of logarithms. Liouville had shown in 1840, directly from the defining series, that in fact neither e nor e^2 could be rational or quadratic irrational; but Hermite's work began a new era. In particular, within a decade, Lindemann succeeded in generalizing Hermite's method and, in a classical paper, he proved that π is transcendental and solved thereby the ancient Greek problem concerning the quadrature of the circle. The work of Hermite and Lindemann was simplified by Weierstrass in 1885, and further simplified by Hilbert, Hurwitz and Gordan in 1893. In [4], the transcendence of e was first proved by Hermite in 1873 by using very different ideas and applying the approximation of analytic functions by rational functions.

Theorem 1.3 (C. Hermite, 1873)**.** *The number e is transcendental number.*

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Theorem 1.4 (F. Lindemann, 1882). *The number* π *is transcendental number.*

In [4], Lindemann stated more general results. One of them is Hermite-Lindemann Theorem:

Theorem 1.5 (Hermite-Lindemann). *If* β *is a non-zero complex number. Then at least one of the two numbers* β *and* e^{β} *is transcendental.*

Thus, if β is algebraic, then e^{β} is transcendental number. Let α be non-zero algebraic number, and if λ is any non-zero determination of its logarithm, then λ is a transcendental number. Now, we define the set $\mathcal{L}$ of logarithm of non-zero algebraic numbers, that is the inverse image of the multiplicative group $\overline{\mathbb{Q}}^{\times}$ by the exponential map :

$$\mathcal{L} = \exp^{-1}(\overline{\mathbb{Q}}^{\times}) = \{\lambda \in \mathbb{C} : e^{\lambda} \in \overline{\mathbb{Q}}^{\times}\}.$$

The theorem of Hermite- Lindemann can be written $\overline{\mathbb{Q}} \cap \mathcal{L} = \{0\}$, that is, $\lambda\,(\neq 0) \in \mathcal{L}$ is transcendental number.

Theorem 1.6 (Lindemann-Weierstrass, 1885). *If* $\beta_1, \ldots, \beta_n$ *are distinct algebraic numbers, then* $e^{\beta_1}, \ldots, e^{\beta_n}$ *are linearly independent over* $\overline{\mathbb{Q}}$.

In 1900, at the International Congress of Mathematicians held in Paris, Hilbert raised, as the seventh of his famous list of 23 problems, the question whether an irrational logarithms of an algebraic number to an algebraic base is transcendental. The question is capable of various alternative formulations; thus one can ask whether an irrational quotient of natural logarithms of algebraic number is transcendental, or whether α^{β} is transcendental for any algebraic number $\alpha \neq 0, 1$ and any algebraic irrational β.

Theorem 1.7 (Gelfond-Schneider, 1934). *Suppose that* $\alpha \neq 0, 1$ *and that* β *is irrational. Then* α, β *and* α^{β} *cannot all be algebraic.*

In particular, $2^{\sqrt{2}}$ and $e^{\pi} = (-1)^{-i}$ are transcendental numbers. In the same year, Gelfond published extended his results [5] of the Gelfond-Schneider Theorem without proof.

Gelfond was the first to study algebraic independence of the values of the exponential function at points that are not necessarily algebraic. In 1948, he conjectured that *if* $\alpha, \beta \in \overline{\mathbb{Q}}$, $\alpha \neq 0, 1$, $\deg \beta = d \geq 2$, *then* $\alpha^{\beta}, \alpha^{\beta^2}, \ldots, \alpha^{\beta^{d-1}}$ *are algebraically independent.* In general this conjecture is still open. We will discuss it later. Gelfond proved the conjecture for $d = 3$ in 1948. The following result is more general than Gelfond's.

Theorem 1.8 (R. Tijdemann, 1971). *Let* p, q *be positive integers with* $\frac{pq+p}{p+q} \geq 2$. *Let* $\{a_1, \ldots, a_p\}$ *and* $\{b_1, \ldots, b_q\}$ *be two sets of* $\mathbb{Q}$*-linearly independent complex numbers. Then the transcendence degree of*

$$\mathbb{Q}(a_1, \ldots, a_p, e^{a_1 b_1}, \ldots, e^{a_p b_q}) \geq 2.$$

In 1949, Gelfond proved Theorem 5.1 for the case $p = q = 3$ with some conditions on the numbers a_i, b_j for $1 \leq i \leq p$, $1 \leq j \leq q$. Theorem 5.1 in the present general form was proved by Tijdemann in 1971. We derive some of consequences

Theorem 1.9 (A. Gelfond, 1948). *Let* $\alpha, \beta \in \overline{\mathbb{Q}}$ *with* $\alpha \neq 0, 1$ *and* $\deg \beta = 3$. *Then* $\alpha^{\beta}, \alpha^{\beta^2}$ *are algebraically independent.*

Proof. Take $p = q = 3$, $a_j = \beta^{j-1}$, $b_j = \beta^{j-1}\log\alpha$ for $j = 1,2,3$. Since $\deg\beta = 3$, all the numbers $\beta^j, \alpha^{\beta^j}$ for $j \geq 1$ are algebraic over $\mathbb{Q}(\alpha^\beta, \alpha^{\beta^2})$. Hence by Theorem 5.1, α^β and α^{β^2} are algebraically independent. □

Theorem 1.10 (Shmelev, 1968)**.** *Let* $\alpha_1, \alpha_2 \in \overline{\mathbb{Q}}$ *such that* $\log\alpha_1$ *and* $\log\alpha_2$ *are linearly independent over* $\mathbb{Q}$*. Suppose* $\beta \in \overline{\mathbb{Q}}$ *with* $\deg\beta = 2$*. Then at least two of the numbers* $\frac{\log\alpha_2}{\log\alpha_1}, \alpha_1^\beta, \alpha_2^\beta$ *are algebraically independent.*

Proof. We take $p = 4$, $q = 2$, $\gamma = \frac{\log\alpha_2}{\log\alpha_1}$, $a_1 = 1$, $a_2 = \gamma$, $a_3 = \beta$, $a_4 = \beta\gamma$, $b_1 = \log\alpha_1$, $b_2 = \beta\log\alpha_1$. Then we see that $e^{a_i b_j}$ for $1 \leq i \leq 4$, $1 \leq j \leq 2$ are algebraic over $\mathbb{Q}(\gamma, \alpha_1^\beta, \alpha_2^\beta)$. Now the result follows from Theorem 1.8. □

2. Gelfond's Conjectures in CRAS 1934

In [5,6], A.O. Gelfond made an attempt in a one page note in the *Comptes rendus hebdomadires des séances de l'Acadḿies des Scineces de Paris séance, du 23 juillet 1934, Weekly Proceeding of the French Academy of Sciences in 23rd July 1934,* just after he solved the 7th problem of Hilbert on the transcendence of α^β. Now, we translate the proceeding of the CRAS in 1934 written by Gelfond.

I have shown that the number ω^r, where $\omega \neq 0,1$ is algebraic number, r is an irrational algebraic number, must be transcendental. By a generalization of the method stated above theorem, I have shown that the following more general results.

Conjecture 2.1. *Let* $P(x_1, x_2, \ldots, x_n, y_1, \ldots, y_m)$ *be a polynomial with rational coefficients and* $\alpha_1, \alpha_2, \ldots, \alpha_n, \beta_1, \ldots, \beta_m$ *algebraic numbers,* $\beta_i \neq 0, 1$.
The equality

$$P(e^{\alpha_1}, e^{\alpha_2}, \ldots, e^{\alpha_n}, \log\beta_1, \log\beta_2, \ldots, \log\beta_n) = 0$$

is impossible; the numbers $\alpha_1, \alpha_2, \ldots, \alpha_n$*, as well as the numbers* $\log\beta_1, \log\beta_2, \ldots, \log\beta_m$ *are linearly independent over* $\mathbb{Q}$*.*

This theorem includes as special cases, the theorem of Hermite and Lindemann, the complete solutions of Hilbert's problem, the transcendence of the number $e^{\omega_1 e^{\omega_2}}$ (where ω_1 and ω_2 are algebraic numbers), the theorem on the relative transcendence of the numbers e and π.

Conjecture 2.2. *The numbers*

$$e^{\omega_1 e^{\omega_2 e^{\cdot^{\cdot^{\cdot \omega_{n-1} e^{\omega_n}}}}}} \quad \text{and} \quad \alpha_1^{\alpha_2^{\alpha_3^{\cdot^{\cdot^{\cdot \alpha_m}}}}}$$

where $\omega_1 \neq 0, \omega_2, \ldots, \omega_n$ *et* $\alpha_1 \neq 0, 1$, $\alpha_2 \neq 0, 1$, $\alpha_3 \neq 0$, $\alpha_4, \ldots, \alpha_m$ *are algebraic numbers, are transcendental numbers, and among numbers of this form there is n non-trivial algebraic relations with rational integer coefficients.*

Conjecture 2.3 (Schanuel's Conjecture)**.** *If* $\alpha, \ldots, \alpha_n \in \mathbb{C}$ *are linearly independent over* $\mathbb{Q}$*, then the set* $\{\alpha_1, \ldots, \alpha_n, e^{\alpha_1}, \ldots, e^{\alpha_n}\}$ *contains at least n algebraically independent numbers, i.e.,*

$$\operatorname{tr\,deg}_{\mathbb{Q}} \mathbb{Q}\Big(\alpha_1, \ldots, \alpha_n, e^{\alpha_1}, \ldots, e^{\alpha_n}\Big) \geq n.$$

When the case $n = 1$ Schanuel's conjecture is true by Hermite-Lindemann Theorem, which stats that e^{α} is transcendental if $\alpha \in \overline{\mathbb{Q}}^{\times}$. Now, we introduce the new definition of the dependent of two sets for introduce the weak version conjecture of Schanuel.

Definition 2.4. *A set $X \subset \mathbb{C}$ is $\overline{\mathbb{Q}}$-dependent on a set $Y \subset \mathbb{C}$ if $\overline{\mathbb{Q}}(X) \subset \overline{\mathbb{Q}}(Y)$*

Note that the fields

$$\mathbb{Q}(\beta_1, \ldots, \beta_n) \;\&\; \mathbb{Q}\Big(\alpha_1, \ldots, \alpha_n, e^{\alpha_1}, \ldots, e^{\alpha_n}\Big)$$

have the same algebraic closure. They also have the same transcendence degree.

Conjecture 2.5 ((WSC) Weak Schnauel's Conjecture)**.** *Given $\alpha_1, \ldots, \alpha_n \in \mathbb{C}$ are linearly independent over $\mathbb{Q}$, if the set $\{\alpha_1, \ldots, \alpha_n, e^{\alpha_1}, \ldots, e^{\alpha_n}\}$ is $\overline{\mathbb{Q}}$-dependent on a subset $\{\beta_1, \ldots, \beta_n\}$ then the numbers $\beta_1, \ldots, \beta_n$ are algebraically independent.*

Now, if Schanuel's conjecture is true, then we have

$$\operatorname{tr}\deg_{\mathbb{Q}} \mathbb{Q}(\beta_1, \ldots, \beta_n) = \operatorname{tr}\deg_{\mathbb{Q}} \mathbb{Q}\Big(\alpha_1, \ldots, \alpha_n, e^{\alpha_1}, \ldots, e^{\alpha_n}\Big) \geq n,$$

and so $\beta_1, \ldots, \beta_n$ are algebraically independent.

Therefore, Schnauel's Conjcture implies Weak Schnauel's Conjecture. We formulate Gelfond's second extension as a conjecture and quote it verbatim, including his partial italicization and his omission of the hypothesis that the α_k are irrational.

Conjecture 2.6 (Gelfond's Power Tower Conjecture)**.** *Let $\omega \neq 0$ and α be algebraic numbers, with α irrational. Then when $z := e^{\omega}$ and when $z := \alpha$, the power tower of z of order $k \geq 2$*

$${}^{k}z = \underbrace{z^{z^{\cdot^{\cdot^{\cdot^{z}}}}}}_{k-times}$$

is transcendental. In fact, when $z := e^{\omega}$ the number ${}^{1}z(=z)$, ${}^{2}z(=z^{z})$, ${}^{3}z(=z^{z^{z}})$, $\ldots$ are algebraically independent, as one ${}^{2}z$, ${}^{3}z$, ${}^{4}z$, $\ldots$ when $z := \alpha$.

Theorem 2.7. *Assume the Weak Schanuel Conjecture. The Gelfond's Power Tower Conjecture is also true. Moreover, under the weaker hypothesis that α is an algebraic number but not a rational integer, the power tower ${}^{m}\alpha$ of order $m \geq 3$ is transcendental and the numbers $\log\alpha$, ${}^{3}\alpha$, ${}^{4}\alpha$, ${}^{5}\alpha$, $\ldots$ are algebraically independent.*

Remark. For example, take $\omega = 1$ and $\alpha = \frac{1}{2}$. Then if WSC holds, the numbers

$$e,\ e^{e},\ e^{e^{e}},\ \ldots$$

are algebraically independent, and so are numbers

$$\log 2,\ \left(\frac{1}{2}\right)^{\frac{1}{\sqrt{2}}},\ \left(\frac{1}{2}\right)^{(\frac{1}{2})^{\frac{1}{\sqrt{2}}}},\ \left(\frac{1}{2}\right)^{(\frac{1}{2})^{(\frac{1}{2})^{\frac{1}{\sqrt{2}}}}},\ \ldots.$$

Proof. Proof for $z := e^{\omega}$. As $0 \neq \omega \in \overline{\mathbb{Q}}$ the Hermite-Lindemann theorem implies that ${}^{1}z = z = e^{\omega}$ is transcendental, and so the statement is true for $k = 1$. Now, fix $k > 0$ and suppose inductively that the numbers ${}^{1}z$, ${}^{2}z, \ldots,$ ${}^{k}z$ are algebraically independent

with $z = e^{\omega}$. Then $\omega,\ \omega \cdot {}^{1}z,\ \omega \cdot {}^{2}z, \ldots,\ \omega \cdot {}^{k}z$ are $\mathbb{Q}$-linearly independent.
As ${}^{j+1}z = e^{\omega \cdot {}^{j}z}$, WSC applied to the subset

$$\left\{ {}^{1}z,\ {}^{2}z, \ldots,\ {}^{k+1}z \right\} \subset \left\{ \omega,\ \omega \cdot {}^{1}z,\ \omega \cdot {}^{2}z, \ldots,\ \omega \cdot {}^{k}z \right. \\ \left. e^{\omega \cdot {}^{2}z}, \ldots,\ e^{\omega \cdot {}^{k}z} \right\}$$

yields the algebraic independence of the numbers ${}^{1}z, {}^{2}z, \ldots,\ {}^{k+1}z$. This completes the induction and proves the theorem for $z := e^{\omega}$.

Proof for $z := \alpha$. Assuming WSC is true, we show that if $\alpha \in \overline{\mathbb{Q}} \backslash \mathbb{Z}$ and $m \geq 3$, then the numbers $\log\alpha, {}^{3}\alpha, {}^{4}\alpha, \ldots, {}^{m}\alpha$ are algebraically independent; the proof in two cases.

Case 1. $\alpha \in \mathbb{Q} \backslash \mathbb{Z}$. Then α^{α} is irrational. Thus 1 and α are $\mathbb{Q}$-linearly independent, and then so one $\log\alpha$ and $\alpha^{\alpha}\log\alpha$. Since α and α^{α} are algebraic, WSC applied to the subset

$$\left\{ \log\alpha,\ \alpha^{\alpha^{\alpha}} \right\} \subset \left\{ \log\alpha,\ \alpha^{\alpha}\log\alpha,\ \alpha,\ \alpha^{\alpha^{\alpha}} \right\}$$

yields the algebraic independence of $\log\alpha$ and $\alpha^{\alpha^{\alpha}} = {}^{3}\alpha$.

Now, fix $m > 3$ and assume inductively that $\log\alpha, {}^{3}\alpha, {}^{4}\alpha, \ldots, {}^{m-1}\alpha$ are algebraically independent. Then in any $\mathbb{Q}$-linear relation

$$\sum_{j=1}^{m-1} a_j\, {}^{j}\alpha = 0 \quad \text{we must have } a_3 = \cdots = a_{m-1} = 0$$

Since ${}^{1}\alpha = \alpha \in \mathbb{Q} \backslash \mathbb{Z}$ and ${}^{2}\alpha = \alpha^{\alpha} \notin \mathbb{Q}$, we have $a_1 = a_2 = 0$. That implies the $\mathbb{Q}$-linear independence of

$$\left\{ {}^{1}\alpha\log\alpha, \ldots, {}^{m-1}\alpha\log\alpha \right\} = \left\{ \log\left({}^{2}\alpha\right), \ldots,\ \log\left({}^{m}\alpha\right) \right\}$$

Then WSC yields the algebraic independence of the subset

$$\left\{ \log\left({}^{2}\alpha\right),\ {}^{3}\alpha,\ {}^{4}\alpha, \ldots, {}^{m}\alpha \right\} \subset \left\{ \log\left({}^{2}\alpha\right), \ldots,\ \log\left({}^{m}\alpha\right), \right. \\ \left. {}^{2}\alpha, \ldots,\ {}^{m}\alpha \right\}$$

and hence, since $\log\left({}^{2}\alpha\right) = \alpha\log\alpha$, also that of the set $\left\{ \log\alpha,\ {}^{3}\alpha,\ {}^{4}\alpha, \ldots,\ {}^{m}\alpha \right\}$. This completes the induction.

Case 2. $\alpha \in \overline{\mathbb{Q}} \backslash \mathbb{Q}$. By the Gelfond-Schneider Theorem, α^{α} is transcendental. Hence $1,\ \alpha,\ \alpha^{\alpha}$ are $\mathbb{Q}$-linearly independent, and then so are $\log\alpha,\ \alpha\log\alpha,\ \alpha^{\alpha}\log\alpha$. Since

$$\left\{ \alpha\log\alpha,\ \alpha^{\alpha}\log\alpha \right\} \subset \overline{\mathbb{Q}}\left(\log\alpha,\ \alpha,\ \alpha^{\alpha} \right)$$

and α is algebraic, WSC applied to the subset

$$\left\{ \log\alpha, \alpha^{\alpha}, \alpha^{\alpha^{\alpha}} \right\} \subset \left\{ \log\alpha, \alpha\log\alpha, \alpha^{\alpha}\log\alpha, \alpha, \alpha^{\alpha}, \alpha^{\alpha^{\alpha}} \right\}$$

yields the algebraic independence of

$$\left\{ \log\alpha,\ \alpha^{\alpha},\ \alpha^{\alpha^{\alpha}} \right\} = \left\{ \log\alpha, {}^{2}\alpha, {}^{3}\alpha \right\}$$

Suppose inductively that $\log\alpha,\ {}^2\alpha,\ {}^3\alpha,\dots,\ {}^{m-1}\alpha$ are algebraically independent, where $m>3$. Then any $\mathbb{Q}$-linear relation

$$a_0+\sum_{j=1}^{m-1}a_j\,{}^j\alpha=0 \quad \text{implies} \quad a_2=\cdots=a_{m-1}=0.$$

Since ${}^1\alpha=\alpha\notin\mathbb{Q}$, we also get $a_0=a_1=0$. That implies the $\mathbb{Q}$-linear independence of

$$\left\{\log\alpha,\ {}^1\alpha\log\alpha,\dots,\ {}^{m-1}\alpha\log\alpha\right\}=\left\{\log\alpha,\ \log\left({}^2\alpha\right),\dots,\ \log\left({}^m\alpha\right)\right\}$$

Since

$$\left\{{}^1\alpha\log\alpha,\dots,\ {}^{m-1}\alpha\log\alpha\right\}\subset\overline{\mathbb{Q}}\left(\log\alpha,\ {}^1\alpha,\ {}^2\alpha,\dots,{}^m\alpha\right),$$

we may apply Weak Schanuel Conjecture

$$\left\{\log\alpha,{}^2\alpha,{}^3\alpha,\dots,{}^m\alpha\right\}\subset\left\{\log\left({}^1\alpha\right),\dots,\log\left({}^m\alpha\right),{}^1\alpha,\dots,{}^m\alpha\right\}$$

and conclude that $\log\alpha,{}^2\alpha,{}^3\alpha,\dots,\ {}^m\alpha$ are algebraically independent. Thus, in both Cases 1 and 2, the numbers $\log\alpha,\ {}^3\alpha,\ {}^4\alpha,\dots,\ {}^m\alpha$ are algebraically independent. □

Results on the arithmetic nature of power towers of x of infinite order

$${}^\infty x:=\lim_{k\to\infty}{}^k x=x^{x^{x^{\cdot^{\cdot^{\cdot}}}}} \qquad (e^{-e}\le x\le e^{\frac{1}{e}}).$$

Theorem 2.8. *If the Weak Schanuel Conjecture is true, then for any non-constant polynomials $P(x),Q(x)\in\mathbb{Q}[x]$, the numbers $P(e)^{Q(e)}$, $P(\pi)^{Q(\pi)}$ are transcendental.*

- Our proof can adapted to show that WSC also implies that the transcendence of $P(\log 2)^{Q(\log 2)}$.
- On the other hand, there do exist transcendental numbers T for which T^T is algebraic.
- In the view of the Gelfond-Schneider, it is natural to ask
- Which transcendental numbers are not algebraic powers of algebraic numbers?
- For instance, $e=\alpha^\beta$ for any $\alpha,\beta\in\overline{\mathbb{Q}}$, since otherwise $e^{\frac{1}{\beta}}=\alpha\in\overline{\mathbb{Q}}$ would contradict the Hermite-Lindemann Theorem

Proof. Fix non-constant polynomials $P(x),Q(x)\in\overline{\mathbb{Q}}[x]$.

Proof that $P(e)^{Q(e)}$ is transcendental. Firstly, let us consider the cases $P(x)=x^n$, where $n\ge1$. Since $Q(e)$ is transcendental, 1 and $nQ(e)$ are $\mathbb{Q}$-linearly independent. Applying WSC to the subset

$$\left\{e,e^{nQ(e)}\right\}\subset\left\{1,nQ(e),e,e^{nQ(e)}\right\}$$

it follows that $e^{nQ(e)}=P(e)^{Q(e)}$ is transcendental, as claimed.

Now, assume $P(x)\ne x^n$ for any $n\ge1$. We show that 1 and $\log P(e)$ are $\mathbb{Q}$-linearly independent. Given a $\mathbb{Q}$-linear relation $a+b\log P(e)=0$ by clearing the denominators if necessary, we may assume that $a,b\in\mathbb{Z}$ with $b\ge0$. Now $P(e)^b e^a-1=0$.

If $a \geq 0$, then, since $P(x) \neq x^n$ for any $n \geq 0$ and e is not algebraic the polynomial $P(x)^b x^a - 1$ must be identically zero; hence $= a = b = 0$.

If $a < 0$, then $P(x)^b - x^{-a} = x^{-a}\left(P(x)^b x^a - 1\right)$ must be the zero polynomial, and again $a = b = 0$.

Now, WSC applied to the subset

$$\{e, \log P(e)\} \subset \{1, \log P(e), e, P(e)\}$$

implies that e and $\log P(e)$ are algebraically independent. Hence so are $P(e)$ and $Q(e)$.

Proof that $P(\pi)^{Q(\pi)}$ is transcendental. Note that πi and $\log P(\pi)$ are $\mathbb{Q}$-linearly independent, for if there exists a $\mathbb{Z}$-relation $ai\pi + b\log P(\pi) = 0$ with $b > 0$, then $P(\pi)^b = (-1)^a$ would be algebraic, contradicting the transcendence of π. Applying Weak Schanuel's Conjecture to the subset

$$\{i\pi,\ \log P(\pi)\} \subset \{i\pi,\ \log P(\pi),\ e^{\pi i},\ P(\pi)\}$$

we get that $i\pi$ and $\log P(\pi)$ are algebraically independent.
Then the set

$$\{i\pi,\ \log P(\pi),\ Q(\pi),\ \log P(\pi)\}$$

is $\mathbb{Q}$-linearly independent, and WSC applied to this subset of

$$\left\{i\pi,\ \log P(\pi),\ Q(\pi)\log P(\pi),\ e^{\pi i},\ P(\pi),\ P(\pi)^{Q(\pi)}\right\}$$

yields the desired result. □

Theorem 2.9. *Assume that the Schanuel Subset Conjecture is true. Let α and β be any algebraic numbers, and let $P(x) \in \overline{\mathbb{Q}}[x]$ be non-constant polynomials. Then*

$$\left(\alpha^\beta - P(e)\right)\left(\alpha^\beta - P(\pi)\right)\left(\alpha^\beta - P(\log 2)\right) \neq 0.$$

Proof. It suffices to show that if Weak Schanuel's Conjecture is true and $\omega \neq 0$ is algebraic, then the numbers $P(e)^\omega$, $P(\pi)^\omega$, and $P(\log 2)^\omega$ are all transcendental.

Lemma 2.10. *If $Q(x) \in \overline{\mathbb{Q}}[x] \backslash \{0, x^n\ :\ n = 0, 1, 2, \ldots\}$, then $\log Q(e)$ is transcendental.*

Proof. Suppose on the contrary that $\alpha := \log Q(e) \in \overline{\mathbb{Q}}$.
If $Q(x) = \sum_{k=0}^{n} a_k x^k$ then we have the relation

$$a_0 + a_1 e + \cdots + a_n e^n - e^\alpha = 0.$$

The Lindemann-Weierstras Theorem implies first that $\alpha \in \{0, 1, \ldots, n\}$ and then that $a_\alpha = 1$ and $a_k = 0$ for $k \neq \alpha$. But then $Q(x) = x^\alpha$ contradicting the hypothesis. Therefore $\log Q(e) \notin \overline{\mathbb{Q}}$. □

Proof that $P(e)^\omega \notin \overline{\mathbb{Q}}$. Since $e^{n\omega}$ is transcendental for $n = 1, 2, \ldots$, we may assume $P(x) \neq x^n$. Then Lemma 1 implies $\log P(e)$ is transcendental, which in turn implies the $\mathbb{Q}$-linear independence of $1,\ \log P(e),\ \omega\log P(e)$. Now, by WSC applied to the subset

$$\left\{e,\log P(e),P(e)^{\omega}\right\}\subset\left\{1,\log P(e),\omega\log P(e),e,P(e),P(e)^{\omega}\right\}$$

we get, in particular, the transcendence of $P(e)^{\omega}$.

Proof that $P(\pi)^{\omega}\notin\overline{\mathbb{Q}}$. In the second part of the proof of Theorem 2, we proved that $i\pi$ and $\log P(\pi)$ are algebraically independent. Since ω is irrational, the set

$$\left\{i\pi,\log P(\pi),\omega\log P(\pi)\right\}$$

is $\mathbb{Q}$-linearly independent. Now, we can apply WSC to the subset

$$\left\{\pi,\ \log P(\pi),\ P(\pi)^{\omega}\right\}\subset\{i\pi,\ \log P(\pi),\ \omega\log P(\pi),\ e^{i\pi},P(\pi),\ P(\pi)^{\omega}\}$$

and conclude that $P(\pi)^{\omega}$ is transcendental.

Proof that $P(\log 2)^{\omega}\notin\overline{\mathbb{Q}}$. The numbers $\log 2$ and $\log P(\log 2)$ are $\mathbb{Q}$-linearly independent. In fact, any $\mathbb{Q}$-relation $a\log 2+b\log P(\log 2)=0$ implies that $P(\log 2)^{b}=2^{-a}$, and then $a=b=0$ by the transcendence of $P(\log 2)$. By WSC applied to the subset

$$\{\log 2,\ \log P(\log 2)\}\subset\{\log 2,\ \log P(\log 2),\ 2,\ P(\log 2)\},$$

we have that $\log 2,\ \log P(\log 2)$ are actually algebraically independent, and so the set

$$\{\log 2,\log P(\log 2),\omega\log P(\log 2)\}$$

is $\mathbb{Q}$-linearly independent.
Applying by Weak Schanuel's Conjecture, applied to the subset

$$\left\{\log 2,\ \log P(\log 2),\ P(\log 2)^{\omega}\right\}\subset\left\{\log 2,\log P(\log 2),\omega P(\log 2),2,P(\log 2),P(\log 2)^{\omega}\right\}$$

we see that $\log 2,\ \log P(\log 2),\ P(\log 2)^{\omega}$ are algebraically independent. The theorem follows.

□

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